

Parabolic Kazhdan-Lusztig polynomials and Schubert varieties

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Abstract

We shall give a description of the intersection cohomology groups of the Schubert varieties in partial flag manifolds over symmetrizable Kac-Moody Lie algebras in terms of parabolic Kazhdan-Lusztig polynomials introduced by Deodhar.

1 Introduction

For a Coxeter system (W, S) Kazhdan-Lusztig [6], [7] introduced polynomials

$$P_{y,w}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k} q^k \in \mathbb{Z}[q], \quad Q_{y,w}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k} q^k \in \mathbb{Z}[q],$$

called a Kazhdan-Lusztig polynomial and an inverse Kazhdan-Lusztig polynomial respectively. Here, (y, w) is a pair of elements of W such that $y \leq w$ with respect to the Bruhat order. These polynomials play important roles in various aspects of the representation theory of reductive algebraic groups.

In the case W is associated to a symmetrizable Kac-Moody Lie algebra \mathfrak{g} , the polynomials have the following geometric meanings. Let $X = G/B$ be the corresponding flag variety (see Kashiwara [3]), and set $X^w = B^-wB/B$ and $X_w = BwB/B$ for $w \in W$. Here B and B^- are the “Borel subgroups” corresponding to the standard Borel subalgebra \mathfrak{b} and its opposite \mathfrak{b}^- respectively. Then X^w (resp. X_w) is an $\ell(w)$ -codimensional (resp. $\ell(w)$ -dimensional) locally closed subscheme of the infinite-dimensional scheme X .

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Here $\ell(w)$ denotes the length of w as an element of the Coxeter group W . Set $X' = \bigcup_{w \in W} X_w$. Then X' coincides with the flag variety considered by Kac-Peterson [2], Tits [10], et al. Moreover we have

$$X = \bigsqcup_{w \in W} X^w, \quad X' = \bigsqcup_{w \in W} X_w,$$

and

$$\overline{X^w} = \bigsqcup_{y \geq w} X^y, \quad \overline{X_w} = \bigsqcup_{y \leq w} X_y$$

for any $w \in W$.

By Kazhdan-Lusztig [7] we have the following result (see also Kashiwara-Tanisaki [4]).

Theorem 1.1. (i) *Let $w, y \in W$ satisfying $w \leq y$. Then we have*

$$H^{2k+1}(\pi \mathbb{Q}_{X^w}^H)_{yB/B} = 0, \quad H^{2k}(\pi \mathbb{Q}_{X^w}^H)_{yB/B} = \mathbb{Q}^H(-k)^{\oplus Q_{w,y,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module $\pi \mathbb{Q}_{X^y}^H[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{X^w}^H[-\ell(w)]$ coincides with $P_{w,y,k}$.*

Theorem 1.2. (i) *Let $w, y \in W$ satisfying $w \geq y$. Then we have*

$$H^{2k+1}(\pi \mathbb{Q}_{X_w}^H)_{yB/B} = 0, \quad H^{2k}(\pi \mathbb{Q}_{X_w}^H)_{yB/B} = \mathbb{Q}^H(-k)^{\oplus P_{y,w,k}}$$

for any $k \in \mathbb{Z}$.

(ii) *The multiplicity of the irreducible Hodge module $\pi \mathbb{Q}_{X_y}^H[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{X_w}^H[\ell(w)]$ coincides with $Q_{y,w,k}$.*

Here $\pi \mathbb{Q}_{X^w}^H[-\ell(w)]$ and $\pi \mathbb{Q}_{X_w}^H[\ell(w)]$ denote the Hodge modules corresponding to the perverse sheaves $\pi \mathbb{Q}_{X^w}[-\ell(w)]$ and $\pi \mathbb{Q}_{X_w}[\ell(w)]$ respectively. In Theorem 1.1 we have used the convention so that $\pi \mathbb{Q}_Z^H[-\text{codim } Z]$ is a Hodge module for a locally closed finite-codimensional subvariety Z since we deal with sheaves supported on finite-codimensional subvarieties, while in Theorem 1.2 we have used another convention so that $\pi \mathbb{Q}_Z^H[\dim Z]$ is a Hodge modules for a locally closed finite-dimensional subvariety Z since we deal with sheaves supported on finite-dimensional subvarieties.

Let J be a subset of S . Set $W_J = \langle J \rangle$ and denote by W^J the set of elements $w \in W$ whose length is minimal in the coset wW_J . In [1] Deodhar introduced two generalizations of the Kazhdan-Lusztig polynomials to this relative situation. For $(y, w) \in W^J \times W^J$ such that $y \leq w$ we denote the parabolic Kazhdan-Lusztig polynomial for $u = -1$ by

$$P_{y,w}^{J,q}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,q} q^k \in \mathbb{Z}[q],$$

and that for $u = q$ by

$$P_{y,w}^{J,-1}(q) = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,-1} q^k \in \mathbb{Z}[q]$$

contrary to the original reference [1]. We can also define inverse parabolic Kazhdan-Lusztig polynomials

$$Q_{y,w}^{J,q}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k}^{J,q} q^k \in \mathbb{Z}[q], \quad Q_{y,w}^{J,-1}(q) = \sum_{k \in \mathbb{Z}} Q_{y,w,k}^{J,-1} q^k \in \mathbb{Z}[q]$$

(see § 2 below)

The aim of this paper is to extend Theorem 1.1 and Theorem 1.2 to this relative situation using the partial flag variety corresponding to J .

Let Y be the partial flag variety corresponding to J . Let 1_Y be the origin of Y and set $Y^w = B^-w1_Y$ and $Y_w = Bw1_Y$ for $w \in W^J$. Then Y^w (resp. Y_w) is an $\ell(w)$ -codimensional (resp. $\ell(w)$ -dimensional) locally closed subscheme of the infinite-dimensional scheme Y . Set $Y' = \bigcup_{w \in W^J} Y_w$. Then we have

$$Y = \bigsqcup_{w \in W^J} Y^w, \quad Y' = \bigsqcup_{w \in W^J} Y_w,$$

and

$$\overline{Y^w} = \bigsqcup_{y \geq w} Y^y, \quad \overline{Y_w} = \bigsqcup_{y \leq w} Y_y$$

for any $w \in W^J$.

We note that the construction of the partial flag variety similar to the ordinary flag variety in Kashiwara [3] has not yet appeared in the literature. In the case where W_J is a finite group (especially when W is an affine Weyl group), we can construct the partial flag variety $Y = G/P$ and the properties of Schubert varieties in Y stated above are established in exactly the same manner as in Kashiwara [3] and Kashiwara-Tanisaki [5]. In the case W_J is an

infinite group we can not define the “parabolic subgroup” P corresponding to J as a group scheme and hence the arguments in Kashiwara [3] are not directly generalized. We leave the necessary modification in the case W_J is an infinite group to the future work.

Our main result is the following.

Theorem 1.3. (i) *Let $w, y \in W^J$ satisfying $w \leq y$. Then we have*

$$H^{2k+1}(\pi \mathbb{Q}_{Y_w}^H)_{y1_Y} = 0, \quad H^{2k}(\pi \mathbb{Q}_{Y_w}^H)_{y1_Y} = \mathbb{Q}^H(-k)^{\oplus Q_{w,y,k}^{J,-1}}$$

for any $k \in \mathbb{Z}$.

- (ii) *The multiplicity of the irreducible Hodge module $\pi \mathbb{Q}_{Y_y}^H[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{Y_w}^H[-\ell(w)]$ coincides with $P_{w,y,k}^{J,-1}$.*

Theorem 1.4. (i) *Let $w, y \in W^J$ satisfying $w \geq y$. Then we have*

$$H^{2k+1}(\pi \mathbb{Q}_{Y_w}^H)_{y1_Y} = 0, \quad H^{2k}(\pi \mathbb{Q}_{Y_w}^H)_{y1_Y} = \mathbb{Q}^H(-k)^{\oplus P_{y,w,k}^{J,q}}$$

for any $k \in \mathbb{Z}$.

- (ii) *The multiplicity of the irreducible Hodge module $\pi \mathbb{Q}_{Y_y}^H[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{Y_w}^H[\ell(w)]$ coincides with $Q_{y,w,k}^{J,-1}$.*

In Theorem 1.3 we have used the convention so that $\pi \mathbb{Q}_Z^H[-\text{codim } Z]$ is a Hodge module for a locally closed finite-codimensional subvariety Z , and in Theorem 1.4 we have used another convention so that $\pi \mathbb{Q}_Z^H[\dim Z]$ is a Hodge modules for a locally closed finite-dimensional subvariety Z .

We note that a result closely related to Theorem 1.4 was already obtained by Deodhar [1].

The above results imply that the coefficients of the four (ordinary or inverse) parabolic Kazhdan-Lusztig polynomials are all non-negative in the case W is the Weyl group of a symmetrizable Kac-Moody Lie algebra.

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2 Kazhdan-Lusztig polynomials

Let R be a commutative ring containing $\mathbb{Z}[q, q^{-1}]$ equipped with a direct sum decomposition $R = \bigoplus_{k \in \mathbb{Z}} R_k$ into \mathbb{Z} -submodules and an involutive ring

endomorphism $R \ni r \mapsto \bar{r} \in R$ satisfying the following conditions:

$$(2.1) \quad R_i R_j \subset R_{i+j}, \quad \overline{R_i} = R_{-i}, \quad 1 \in R_0, \quad q \in R_2, \quad \bar{q} = q^{-1}.$$

Let (W, S) be a Coxeter system. We denote by $\ell : W \rightarrow \mathbb{Z}_{\geq 0}$ and \geq the length function and the Bruhat order respectively. The Hecke algebra $H = H(W)$ over R is an R -algebra with free R -basis $\{T_w\}_{w \in W}$ whose multiplication is determined by the following:

$$(2.2) \quad T_{w_1} T_{w_2} = T_{w_1 w_2} \quad \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2),$$

$$(2.3) \quad (T_s + 1)(T_s - q) = 0 \quad \text{for } s \in S.$$

Note that $T_e = 1$ by (2.2).

We define involutive ring endomorphisms $H \ni h \mapsto \bar{h} \in H$ and $j : H \rightarrow H$ by

$$(2.4) \quad \overline{\sum_{w \in W} r_w T_w} = \sum_{w \in W} \bar{r}_w T_{w^{-1}}^{-1}, \quad j\left(\sum_{w \in W} r_w T_w\right) = \sum_{w \in W} r_w (-q)^{\ell(w)} T_{w^{-1}}^{-1}.$$

Note that j is an endomorphism of an R -algebra.

Proposition 2.1 (Kazhdan-Lusztig [6]). *For any $w \in W$ there exists a unique $C_w \in H$ satisfying the following conditions:*

$$(2.5) \quad C_w = \sum_{y \leq w} P_{y,w} T_y \quad \text{with } P_{w,w} = 1 \quad \text{and } P_{y,w} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i$$

for $y < w$,

$$(2.6) \quad \overline{C_w} = q^{-\ell(w)} C_w.$$

Moreover we have $P_{y,w} \in \mathbb{Z}[q]$ for any $y \leq w$.

Note that $\{C_w\}_{w \in W}$ is a basis of the R -module H . The polynomials $P_{y,w}$ for $y \leq w$ are called Kazhdan-Lusztig polynomials. We write

$$(2.7) \quad P_{y,w} = \sum_{k \in \mathbb{Z}} P_{y,w,k} q^k.$$

Set $H^* = H^*(W) = \text{Hom}_R(H, R)$. We denote by $\langle \cdot, \cdot \rangle$ the coupling between H^* and H . We define involutions $H^* \ni m \mapsto \bar{m} \in H^*$ and $j : H^* \rightarrow H^*$ by

$$(2.8) \quad \langle \bar{m}, h \rangle = \overline{\langle m, \bar{h} \rangle}, \quad \langle j(m), h \rangle = \langle m, j(h) \rangle \quad \text{for } m \in H^* \text{ and } h \in H.$$

Note that j is an endomorphism of an R -module. For $w \in W$ we define elements $S_w, D_w \in H^*$ by

$$(2.9) \quad \langle S_w, T_x \rangle = (-1)^{\ell(w)} \delta_{w,x}, \quad \langle D_w, C_x \rangle = (-1)^{\ell(w)} \delta_{w,x}.$$

Then any element of H^* is uniquely written as an infinite sum in two ways $\sum_{w \in W} r_w S_w$ and $\sum_{w \in W} r'_w D_w$ with $r_w, r'_w \in R$. Note that we have

$$(2.10) \quad S_w = \sum_{y \geq w} (-1)^{\ell(w) - \ell(y)} P_{w,y} D_y$$

by $C_w = \sum_{y \leq w} P_{y,w} T_y$. By (2.6), we have

$$(2.11) \quad \overline{D}_w = q^{\ell(w)} D_w,$$

and we can write

$$(2.12) \quad D_w = \sum_{y \geq w} Q_{w,y} S_y,$$

where $Q_{w,y}$ are determined by

$$(2.13) \quad \sum_{w \leq y \leq z} (-1)^{\ell(y) - \ell(w)} Q_{w,y} P_{y,z} = \delta_{w,z}.$$

Note that (2.12) is equivalent to

$$(2.14) \quad T_w = \sum_{y \leq w} (-1)^{\ell(w) - \ell(y)} Q_{y,w} C_y.$$

By (2.13) we see easily that

$$(2.15) \quad Q_{w,y} \in \mathbb{Z}[q],$$

$$(2.16) \quad Q_{w,w} = 1 \text{ and } \deg Q_{w,y} \leq (\ell(y) - \ell(w) - 1)/2 \text{ for } w < y.$$

The polynomials $Q_{w,y}$ for $w \leq y$ are called inverse Kazhdan-Lusztig polynomials (see Kazhdan-Lusztig [7]). We write

$$(2.17) \quad Q_{w,y} = \sum_{k \in \mathbb{Z}} Q_{w,y,k} q^k.$$

The following result is proved similarly to Proposition 2.1 (see Kashiwara-Tanisaki [4]).

Proposition 2.2. *Let $w \in W$. Assume that $D \in H^*$ satisfies the following conditions:*

$$(2.18) \quad D = \sum_{y \geq w} r_y S_y \text{ with } r_w = 1 \text{ and } r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i$$

for $w < y$,

$$(2.19) \quad \overline{D} = q^{\ell(w)} D.$$

Then we have $D = D_w$.

We fix a subset J of S and set

$$(2.20) \quad W_J = \langle J \rangle, \quad W^J = \{w \in W; ws > w \text{ for any } s \in J\}.$$

Then we have

$$(2.21) \quad W = \bigsqcup_{w \in W^J} wW_J,$$

$$(2.22) \quad \ell(wx) = \ell(w) + \ell(x) \text{ for any } w \in W^J \text{ and } x \in W_J.$$

When W_J is a finite group, we denote the longest element of W_J by w_J .

Let $a \in \{q, -1\}$ and define $a^\dagger \in \{q, -1\}$ by $aa^\dagger = -q$. Define an algebra homomorphism $\chi^a : H(W_J) \rightarrow R$ by $\chi^a(T_w) = a^{\ell(w)}$, and denote the corresponding one-dimensional $H(W_J)$ -module by $R^a = R1^a$. We define the induced module $H^{J,a}$ by

$$(2.23) \quad H^{J,a} = H \otimes_{H(W_J)} R^a,$$

and define $\varphi^{J,a} : H \rightarrow H^{J,a}$ by $\varphi^{J,a}(h) = h \otimes 1^a$.

It is easily checked that $H^{J,a} \ni k \mapsto \overline{k} \in H^{J,a}$ and $j^a : H^{J,a} \rightarrow H^{J,a^\dagger}$ are well defined by

$$(2.24) \quad \overline{\varphi^{J,a}(h)} = \varphi^{J,a}(\overline{h}), \quad j^a(\varphi^{J,a}(h)) = \varphi^{J,a^\dagger}(j(h)) \quad \text{for } h \in H.$$

Note that j^a is a homomorphism of R -modules and that

$$(2.25) \quad \overline{rk} = \overline{r}\overline{k} \quad \text{for } r \in R \text{ and } k \in H^{J,a},$$

$$(2.26) \quad \overline{\overline{k}} = k \quad \text{for } k \in H^{J,a},$$

$$(2.27) \quad j^{a^\dagger} \circ j^a = \text{id}_{H^{J,a}}.$$

For $w \in W^J$ set $T_w^{J,a} = \varphi^{J,a}(T_w)$. It is easily seen that $H^{J,a}$ is a free R -module with basis $\{T_w^{J,a}\}_{w \in W^J}$. Note that we have

$$(2.28) \quad \varphi^{J,a}(T_{wx}) = a^{\ell(x)} T_w^{J,a} \quad \text{for } w \in W^J \text{ and } x \in W_J.$$

Proposition 2.3 (Deodhar [1]). *For any $w \in W^J$ there exists a unique $C_w^{J,a} \in H^{J,a}$ satisfying the following conditions.*

$$(2.29) \quad C_w^{J,a} = \sum_{y \leq w} P_{y,w}^{J,a} T_y \text{ with } P_{w,w}^{J,a} = 1 \text{ and } P_{y,w}^{J,a} \in \bigoplus_{i=0}^{\ell(w)-\ell(y)-1} R_i$$

for $y < w$.

$$(2.30) \quad \overline{C_w^{J,a}} = q^{-\ell(w)} C_w^{J,a}.$$

Moreover we have $P_{y,w}^{J,a} \in \mathbb{Z}[q]$ for any $y \leq w$.

The polynomials $P_{y,w}^{J,a}$ for $y, w \in W^J$ with $y \leq w$ are called parabolic Kazhdan-Lusztig polynomials. We write

$$(2.31) \quad P_{y,w}^{J,a} = \sum_{k \in \mathbb{Z}} P_{y,w,k}^{J,a} q^k.$$

Remark 2.4. In the original reference [1] Deodhar uses

$$(-1)^{\ell(w)} j^{a^\dagger} (C_w^{J,a^\dagger}) = \sum_{y \leq w} (-q)^{\ell(w)-\ell(y)} \overline{P_{y,w}^{J,a^\dagger}} T_y^{J,a}$$

instead of $C_w^{J,a}$ to define the parabolic Kazhdan-Lusztig polynomials. Hence our $P_{y,w}^{J,a}$ is actually the parabolic Kazhdan-Lusztig polynomial $P_{y,w}^J$ for $u = a^\dagger$ in the terminology of [1].

Proposition 2.5 (Deodhar [1]). *Let $w, y \in W^J$ such that $w \geq y$.*

(i) *We have*

$$P_{y,w}^{J,-1} = \sum_{x \in W_J, yx \leq w} (-1)^{\ell(x)} P_{yx,w}.$$

(ii) *If W_J is a finite group, then we have $P_{y,w}^{J,q} = P_{yw_J, ww_J}$.*

Set

$$(2.32) \quad H^{J,a,*} = \text{Hom}_R(H^{J,a}, R),$$

and define ${}^t\varphi^{J,a} : H^{J,a,*} \rightarrow H^*$ by

$$\langle {}^t\varphi^{J,a}(n), h \rangle = \langle n, \varphi^{J,a}(h) \rangle \quad \text{for } n \in H^{J,a,*} \text{ and } h \in H.$$

Then ${}^t\varphi^{J,a}$ is an injective homomorphism of R -modules. We define an involution $-$ of $H^{J,a,*}$ similarly to (2.8). We can easily check that

$$(2.33) \quad \overline{{}^t\varphi^{J,a}(n)} = {}^t\varphi^{J,a}(\overline{n}) \quad \text{for any } n \in H^{J,a,*}.$$

For $w \in W^J$ we define $S_w^{J,a}, D_w^{J,a} \in H^{J,a,*}$ by

$$(2.34) \quad \langle S_w^{J,a}, T_x^{J,a} \rangle = (-1)^{\ell(w)} \delta_{w,x}, \quad \langle D_w^{J,a}, C_x^{J,a} \rangle = (-1)^{\ell(w)} \delta_{w,x}.$$

Then any element of $H^{J,a,*}$ is written uniquely as an infinite sum in two ways $\sum_{w \in W^J} r_w S_w^{J,a}$ and $\sum_{w \in W^J} r'_w D_w^{J,a}$ with $r_w, r'_w \in R$. Note that we have

$$(2.35) \quad S_w^{J,a} = \sum_{y \in W^J, y \geq w} (-1)^{\ell(w) - \ell(y)} P_{w,y}^{J,a} D_y^{J,a}$$

by $C_w^{J,a} = \sum_{y \leq w} P_{y,w}^{J,a} T_y$. We see easily by (2.28) that

$$(2.36) \quad {}^t\varphi^{J,a}(S_w^{J,a}) = \sum_{x \in W_J} (-a)^{\ell(x)} S_{wx} \quad \text{for } w \in W^J.$$

By the definition we have

$$(2.37) \quad \overline{D_w^{J,a}} = q^{\ell(w)} D_w^{J,a},$$

and we can write

$$(2.38) \quad D_w^{J,a} = \sum_{y \in W_J, y \geq w} Q_{w,y}^{J,a} S_y^{J,a}$$

where $Q_{w,y}^{J,a} \in R$ are determined by

$$(2.39) \quad \sum_{y \in W^J, w \leq y \leq z} (-1)^{\ell(y) - \ell(w)} Q_{w,y}^{J,a} P_{y,z}^{J,a} = \delta_{w,z}$$

for $w, z \in W^J$ satisfying $w \leq z$.

Note that (2.38) is equivalent to

$$(2.40) \quad T_w^{J,a} = \sum_{y \in W^J, y \leq w} (-1)^{\ell(w) - \ell(y)} Q_{y,w}^{J,a} C_y^{J,a}.$$

By (2.39) we have for $w, y \in W_J$

$$(2.41) \quad Q_{w,y}^{J,a} \in \mathbb{Z}[q],$$

$$(2.42) \quad Q_{w,w}^{J,a} = 1 \text{ and } \deg Q_{w,y}^{J,a} \leq (\ell(y) - \ell(w) - 1)/2 \text{ for } w < y.$$

We call the polynomials $Q_{w,y}^{J,a}$ for $w \leq y$ inverse parabolic Kazhdan-Lusztig polynomials. We write

$$(2.43) \quad Q_{w,y}^{J,a} = \sum_{k \in \mathbb{Z}} Q_{w,y,k}^{J,a} q^k.$$

Similarly to Propositions 2.1, 2.2, 2.3, we can prove the following.

Proposition 2.6. *Let $w \in W^J$. Assume that $D \in H^{J,a,*}$ satisfies the following conditions:*

$$(2.44) \quad D = \sum_{y \in W^J, y \geq w} r_y S_y^{J,a} \text{ with } r_w = 1 \text{ and } r_y \in \bigoplus_{i=0}^{\ell(y)-\ell(w)-1} R_i$$

for $y \in W^J$ satisfying $w < y$.

$$(2.45) \quad \overline{D} = q^{\ell(w)} D.$$

Then we have $D = D_w^{J,a}$.

Proposition 2.7 (Soergel [9]). *Let $w, y \in W^J$ such that $w \leq y$.*

- (i) *We have $Q_{w,y}^{J,-1} = Q_{w,y}$.*
- (ii) *If W_J is a finite group, then we have*

$$Q_{w,y}^{J,q} = \sum_{x \in W_J, ww_J \leq yx} (-1)^{\ell(x)+\ell(w_J)} Q_{ww_J, yx}.$$

3 Hodge modules

In this section we briefly recall the notation from the theory of Hodge modules due to M. Saito [8].

We denote by HS the category of mixed Hodge structures and by HS_k the category of pure Hodge structures with weight $k \in \mathbb{Z}$. Let R and R_k be the Grothendieck groups of HS and HS_k respectively. Then we have $R = \bigoplus_{k \in \mathbb{Z}} R_k$ and R is endowed with a structure of a commutative ring via the tensor product of mixed Hodge structures. The identity element of R is given by $[\mathbb{Q}^H]$, where \mathbb{Q}^H is the trivial Hodge structure. We denote by $R \ni r \mapsto \bar{r} \in R$ the involutive ring endomorphism induced by the duality functor $\mathbb{D} : \text{HS} \rightarrow \text{HS}^{\text{op}}$. Here HS^{op} denotes the opposite category of HS . Let $\mathbb{Q}^H(1)$ and $\mathbb{Q}^H(-1)$ be the Hodge structure of Tate and its dual respectively, and set $\mathbb{Q}^H(\pm n) = \mathbb{Q}^H(\pm 1)^{\otimes n}$ for $n \in \mathbb{Z}_{\geq 0}$. We can regard $\mathbb{Z}[q, q^{-1}]$ as a subring of R by $q^n = [\mathbb{Q}^H(-n)]$. Then the condition (2.1) is satisfied for this R .

Let Z be a finite-dimensional algebraic variety over \mathbb{C} . There are two conventions for perverse sheaves on Z according to whether $\mathbb{Q}_U[\dim U]$ is a perverse sheaf or $\mathbb{Q}_U[-\operatorname{codim} U]$ is a perverse sheaf for a closed smooth subvariety U of Z . Correspondingly, we have two conventions for Hodge modules. When we use the convention so that $\mathbb{Q}_U[\dim U]$ is a perverse sheaf, we denote the category of Hodge modules on Z by $\operatorname{HM}_d(Z)$, and when we use the other one we denote it by $\operatorname{HM}_c(Z)$. Let $D^b(\operatorname{HM}_d(Z))$ and $D^b(\operatorname{HM}_c(Z))$ denote the bounded derived categories of $\operatorname{HM}_d(Z)$ and $\operatorname{HM}_c(Z)$ respectively. Note that d is for dimension and c for codimension. Then the functor $\operatorname{HM}_d(Z) \rightarrow \operatorname{HM}_c(Z)$ given by $M \mapsto M[-\dim Z]$ gives the category equivalences

$$\operatorname{HM}_d(Z) \cong \operatorname{HM}_c(Z), \quad D^b(\operatorname{HM}_d(Z)) \cong D^b(\operatorname{HM}_c(Z)).$$

We shall identify $D^b(\operatorname{HM}_d(Z))$ with $D^b(\operatorname{HM}_c(Z))$ via this equivalence, and then we have

$$(3.1) \quad \operatorname{HM}_c(Z) = \operatorname{HM}_d(Z)[- \dim Z].$$

Although there are no essential differences between $\operatorname{HM}_d(Z)$ and $\operatorname{HM}_c(Z)$, we have to be careful in extending the theory of Hodge modules to the infinite-dimensional situation. In dealing with sheaves supported on finite-dimensional subvarieties embedded into an infinite-dimensional manifold we have to use HM_d , while we need to use HM_c when we treat sheaves supported on finite-codimensional subvariety of an infinite-dimensional manifold. In fact what we really need in the sequel is the results for infinite-dimensional situation; however, we shall only give below a brief explanation for the finite-dimensional case. The extension of HM_d to the infinite-dimensional situation dealing with sheaves supported on finite-dimensional subvarieties is easy, and as for the extension of HM_c to the infinite-dimensional situation dealing with sheaves supported on finite-codimensional subvarieties we refer the readers to Kashiwara-Tanisaki [4].

Let Z be a finite-dimensional algebraic variety over \mathbb{C} . When Z is smooth, one has a Hodge module $\mathbb{Q}_Z^H[\dim Z] \in \operatorname{Ob}(\operatorname{HM}_d(Z))$ corresponding to the perverse sheaf $\mathbb{Q}_Z[\dim Z]$. More generally, for a locally closed smooth subvariety U of Z one has a Hodge module ${}^\pi\mathbb{Q}_U^H[\dim U] \in \operatorname{Ob}(\operatorname{HM}_d(Z))$ corresponding to the perverse sheaf ${}^\pi\mathbb{Q}_U[\dim U]$. For $M \in \operatorname{Ob}(D^b(\operatorname{HM}_d(Z)))$ and $n \in \mathbb{Z}$ we set $M(n) = M \otimes \mathbb{Q}^H(n)$. One has the duality functor

$$(3.2) \quad \mathbb{D}_d : \operatorname{HM}_d(Z) \rightarrow \operatorname{HM}_d(Z)^{\operatorname{op}}, \quad \mathbb{D}_d : D^b(\operatorname{HM}_d(Z)) \rightarrow D^b(\operatorname{HM}_d(Z))^{\operatorname{op}}$$

satisfying $\mathbb{D}_d \circ \mathbb{D}_d = \operatorname{Id}$, and we have

$$(3.3) \quad \mathbb{D}_d({}^\pi\mathbb{Q}_U^H[\dim U]) = {}^\pi\mathbb{Q}_U^H\dim U$$

for a locally closed smooth subvariety U of Z .

Let $f : Z \rightarrow Z'$ be a morphism of finite-dimensional algebraic varieties. Then one has the functors:

$$\begin{aligned} f^* : D^b(\mathrm{HM}_d(Z')) &\rightarrow D^b(\mathrm{HM}_d(Z)), & f^! : D^b(\mathrm{HM}_d(Z')) &\rightarrow D^b(\mathrm{HM}_d(Z)), \\ f_* : D^b(\mathrm{HM}_d(Z)) &\rightarrow D^b(\mathrm{HM}_d(Z')), & f_! : D^b(\mathrm{HM}_d(Z)) &\rightarrow D^b(\mathrm{HM}_d(Z')), \end{aligned}$$

satisfying

$$f^* \circ \mathbb{D}_d = \mathbb{D}_d \circ f^!, \quad f_* \circ \mathbb{D}_d = \mathbb{D}_d \circ f_!.$$

We define the functors $f^*, f^!, f_*, f_!$ for $D^b(\mathrm{HM}_c)$ by identifying $D^b(\mathrm{HM}_c)$ with $D^b(\mathrm{HM}_d)$. For HM_c we use the modified duality functor

$$(3.4) \quad \mathbb{D}_c : \mathrm{HM}_c(Z) \rightarrow \mathrm{HM}_c(Z)^{\mathrm{op}}, \quad \mathbb{D}_c : D^b(\mathrm{HM}_d(Z)) \rightarrow D^b(\mathrm{HM}_d(Z))^{\mathrm{op}}$$

given by

$$\mathbb{D}_c(M) = (\mathbb{D}_d(M))[-2 \dim Z](-\dim Z).$$

It also satisfies $\mathbb{D}_c \circ \mathbb{D}_c = \mathrm{Id}$. For a locally closed smooth subvariety U of Z we have ${}^\pi \mathbb{Q}_U^H[-\mathrm{codim} U] \in \mathrm{Ob}(\mathrm{HM}_c(Z))$ and

$$(3.5) \quad \mathbb{D}_c({}^\pi \mathbb{Q}_U^H[-\mathrm{codim} U]) = {}^\pi \mathbb{Q}_U^H-\mathrm{codim} U.$$

When $f : Z \rightarrow Z'$ is a proper morphism, we have $f_* = f_!$ and hence $f_! \circ \mathbb{D}_d = \mathbb{D}_d \circ f_!$. When f is a smooth morphism, we have $f^! = f^*[2(\dim Z - \dim Z')](\dim Z - \dim Z')$ and hence $f^* \circ \mathbb{D}_c = \mathbb{D}_c \circ f^*$.

4 Finite-codimensional Schubert varieties

Let \mathfrak{g} be a symmetrizable Kac-Moody Lie algebra over \mathbb{C} . We denote by W its Weyl group and by S the set of simple roots. Then (W, S) is a Coxeter system. We shall consider the Hecke algebra $H = H(W)$ over the Grothendieck ring R of the category HS (see § 3), and use the notation in § 2

Let $X = G/B$ be the flag manifold for \mathfrak{g} constructed in Kashiwara [3]. Here B is the “Borel subgroup” corresponding to the standard Borel subalgebra of \mathfrak{g} . Then X is a scheme over \mathbb{C} covered by open subsets isomorphic to

$$\mathbb{A}^\infty = \mathrm{Spec} \mathbb{C}[x_k; k \in \mathbb{N}]$$

(unless $\dim \mathfrak{g} < \infty$).

Let $1_X = eB \in X$ denote the origin of X . For $w \in W$ we have a point $w1_X = wB/B \in X$. Let B^- be the “Borel subgroup” opposite to B , and set $X^w = B^-w1_X = B^-wB/B$ for $w \in W$. Then we have the following result.

Proposition 4.1 (Kashiwara [3]). (i) We have $X = \bigsqcup_{w \in W} X^w$.

(ii) For $w \in W$, X^w is a locally closed subscheme of X isomorphic to \mathbb{A}^∞ (unless $\dim \mathfrak{g} < \infty$) with codimension $\ell(w)$.

(iii) For $w \in W$, we have $\overline{X^w} = \bigsqcup_{y \in W, y \geq w} X^y$.

We call X^w for $w \in W$ a finite-codimensional Schubert cell, and $\overline{X^w}$ a finite-codimensional Schubert variety.

Let J be a subset of S . We denote by Y the partial flag manifold corresponding to J . Let $\pi : X \rightarrow Y$ be the canonical projection and set $1_Y = \pi(1_X)$. We have $\pi(w1_X) = 1_Y$ for any $w \in W_J$. For $w \in W^J$ we set $Y^w = B^-w1_Y = \pi(X^w)$. When W_J is a finite group, we have $Y = G/P_J$ and $Y^w = B^-wP_J/P_J$, where P_J is the “parabolic subgroup” corresponding to J (we cannot define P_J as a group scheme unless W_J is a finite group).

Similarly to Proposition 4.1 we have the following.

Proposition 4.2. (i) We have $Y = \bigsqcup_{w \in W^J} Y^w$.

(ii) For $w \in W^J$, Y^w is a locally closed subscheme of Y isomorphic to \mathbb{A}^∞ (unless $\dim Y < \infty$) with codimension $\ell(w)$.

(iii) For $w \in W^J$, we have $\overline{Y^w} = \bigsqcup_{y \in W^J, y \geq w} Y^y$.

(iv) For $w \in W^J$, we have $\pi^{-1}(Y^w) = \bigsqcup_{x \in W_J} X^{wx}$.

We call a subset Ω of W^J (resp. W) admissible if it satisfies

$$(4.1) \quad w, y \in W^J (\text{resp. } W), w \leq y, y \in \Omega \Rightarrow w \in \Omega.$$

For a finite admissible subset Ω of W^J we set $Y^\Omega = \bigcup_{w \in \Omega} Y^w$. It is a quasi-compact open subset of Y . Let $\text{HM}_c^{B^-}(Y^\Omega)$ be the category of B^- -equivariant Hodge modules on Y^Ω (see Kashiwara-Tanisaki [4] for the equivariant Hodge modules on infinite-dimensional manifolds), and denote its Grothendieck group by $K(\text{HM}_c^{B^-}(Y^\Omega))$. For $w \in W^J$ the Hodge modules $\mathbb{Q}_{Y^w}^H[-\ell(w)]$ and $\pi^*\mathbb{Q}_{Y^w}^H[-\ell(w)]$ are objects of $K(\text{HM}_c^{B^-}(Y^\Omega))$. Note that $\mathbb{Q}_{Y^w}[-\ell(w)]$ is a perverse sheaf on Y because Y^w is affine. Set

$$(4.2) \quad \text{HM}_c^{B^-}(Y) = \varprojlim_{\Omega} \text{HM}_c^{B^-}(Y^\Omega), \quad K(\text{HM}_c^{B^-}(Y)) = \varprojlim_{\Omega} K(\text{HM}_c^{B^-}(Y^\Omega)),$$

where Ω runs through finite admissible subsets of W^J . By the tensor product, $K(\mathrm{HM}_c^{B^-}(Y))$ is endowed with a structure of an R -module. Then any element of $K(\mathrm{HM}_c^{B^-}(Y))$ is uniquely written as an infinite sum

$$\sum_{w \in W^J} r_w [\mathbb{Q}_{Y^w}^H[-\ell(w)]] \text{ with } r_w \in R.$$

Denote by $K(\mathrm{HM}_c^{B^-}(Y)) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_c^{B^-}(Y))$ the involution induced by the duality functor \mathbb{D}_c . Then we have $\overline{r\overline{m}} = \overline{r}m$ for any $r \in R$ and $m \in K(\mathrm{HM}_c^{B^-}(Y))$.

We can similarly define $\mathrm{HM}_c^{B^-}(X)$, $\mathbb{Q}_{X^w}^H[-\ell(w)]$ and ${}^\pi\mathbb{Q}_{X^w}^H[-\ell(w)]$ for $w \in W$, $K(\mathrm{HM}_c^{B^-}(X))$, and $K(\mathrm{HM}_c^{B^-}(X)) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_c^{B^-}(X))$ (for $J = \emptyset$).

Let pt denote the algebraic variety consisting of a single point. For $w \in W$ (resp. $w \in W^J$) we denote by $i_{X,w} : \mathrm{pt} \rightarrow X$ (resp. $i_{Y,w} : \mathrm{pt} \rightarrow Y$) denote the morphism with image $\{w1_X\}$ (resp. $\{w1_Y\}$). We define homomorphisms

$$(4.3) \quad \Phi : K(\mathrm{HM}_c^{B^-}(X)) \rightarrow H^*, \quad \Phi^J : K(\mathrm{HM}_c^{B^-}(Y)) \rightarrow H^{J,-1,*}$$

of R -modules by

$$(4.4) \quad \Phi([M]) = \sum_{w \in W} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{X,w}^*(M)] \right) S_w,$$

$$(4.5) \quad \Phi^J([M]) = \sum_{w \in W^J} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y,w}^*(M)] \right) S_w^{J,-1}.$$

By the definition we have

$$(4.6) \quad \Phi([\mathbb{Q}_{X^w}^H[-\ell(w)]]) = (-1)^{\ell(w)} S_w \quad \text{for } w \in W,$$

$$(4.7) \quad \Phi^J([\mathbb{Q}_{Y^w}^H[-\ell(w)]]) = (-1)^{\ell(w)} S_w^{J,-1} \quad \text{for } w \in W^J,$$

and hence Φ and Φ^J are isomorphisms of R -modules.

The projection $\pi : X \rightarrow Y$ induces a homomorphism

$$\pi^* : K(\mathrm{HM}_c^{B^-}(Y)) \rightarrow K(\mathrm{HM}_c^{B^-}(X))$$

of R -modules.

Lemma 4.3. (i) *The following diagram is commutative.*

$$\begin{array}{ccc} K(\mathrm{HM}_c^{B^-}(Y)) & \xrightarrow{\Phi^J} & H^{J,-1,*} \\ \pi^* \downarrow & & \downarrow {}^t\varphi^{J,-1} \\ K(\mathrm{HM}_c^{B^-}(X)) & \xrightarrow[\Phi]{} & H^* \end{array}$$

(ii) $\overline{\pi^*(m)} = \pi^*(\overline{m})$ for any $m \in K(\mathrm{HM}_c^{B^-}(Y))$.

(iii) $\overline{\Phi(m)} = \Phi(\overline{m})$ for any $m \in K(\mathrm{HM}_c^{B^-}(X))$.

(iv) $\overline{\Phi^J(m)} = \Phi^J(\overline{m})$ for any $m \in K(\mathrm{HM}_c^{B^-}(Y))$.

Proof. For $w \in W^J$ we have $\pi^*(\mathbb{Q}_{Y^w}^H) = \mathbb{Q}_{\pi^{-1}Y^w}^H$, and hence Proposition 4.2 (iv) implies

$$\pi^*([\mathbb{Q}_{Y^w}^H]) = \sum_{x \in W_J} [\mathbb{Q}_{X^{wx}}^H].$$

Thus (i) follows from (4.6), (4.7) and (2.36)

Locally on X the morphism π is a projection of the form $Z \times \mathbb{A}^\infty \rightarrow Z$, and thus $\pi^* \circ \mathbb{D}_c = \mathbb{D}_c \circ \pi^*$. Hence the statement (ii) holds.

The statement (iii) is already known (see Kashiwara-Tanisaki [4]).

Then the statement (iv) follows from (i), (ii), (iii), (2.33) and the injectivity of ${}^t\varphi^{J,-1}$. \square

Theorem 4.4. *Let $w, y \in W^J$ satisfying $w \leq y$. Then we have*

$$H^{2k+1}i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H) = 0, \quad H^{2k}i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H) = \mathbb{Q}^H(-k)^{\oplus Q_{w,y,k}^{J,-1}}$$

for any $k \in \mathbb{Z}$. In particular, we have

$$\Phi^J([\pi\mathbb{Q}_{Y^w}^H[-\ell(w)]]) = (-1)^{\ell(w)} D_w^{J,-1}.$$

Proof. Let $w \in W^J$ and set

$$(-1)^{\ell(w)} \Phi^J([\pi\mathbb{Q}_{Y^w}^H[-\ell(w)]]) = D = \sum_{y \in W^J, y \geq w} r_y S_y^{J,-1}.$$

By the definition of $\pi\mathbb{Q}_{Y^w}^H[-\ell(w)]$ we have

$$\mathbb{D}_c(\pi\mathbb{Q}_{Y^w}^H[-\ell(w)]) = \pi\mathbb{Q}_{Y^w}^H-\ell(w),$$

and hence we obtain

$$(4.8) \quad \overline{D} = q^{\ell(w)} D$$

by Lemma 4.3 (iv). By the definition of Φ^J we have

$$(4.9) \quad r_y = \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H)],$$

and by the definition of $\pi\mathbb{Q}_{Y^w}^H[-\ell(w)]$ we have

$$(4.10) \quad r_w = 1,$$

$$(4.11) \quad \begin{aligned} &\text{for } y > w \text{ we have } H^k i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H) = 0 \text{ unless} \\ &0 \leq k \leq (\ell(y) - \ell(w) - 1). \end{aligned}$$

By the argument similar to Kashiwara-Tanisaki [4] (see also Kazhdan-Lusztig [7]) we have

$$(4.12) \quad [H^k i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H)] \in R_k.$$

In particular, we have

$$(4.13) \quad \text{for } y > w \text{ we have } r_y \in \bigoplus_{k=0}^{\ell(y)-\ell(w)-1} R_k.$$

Thus we obtain $D = D_w^{J,-1}$ by (4.8), (4.10), (4.13) and Proposition 2.6. Hence $r_y = Q_{y,w}^{J,-1}$. By (4.9) and (4.12) we have $[H^{2k+1} i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H)] = 0$ and $[H^{2k} i_{Y,y}^*(\pi\mathbb{Q}_{Y^w}^H)] = q^k Q_{w,y,k}$ for any $k \in \mathbb{Z}$. The proof is complete. \square

By (2.35) and Theorem 4.4 we obtain the following.

Corollary 4.5. *We have*

$$[\mathbb{Q}_{Y^w}^H[-\ell(w)]] = \sum_{y \geq w} P_{w,y}^{J,-1} [\pi\mathbb{Q}_{Y^y}^H[-\ell(y)]]$$

in the Grothendieck group $K(\text{HM}_c^{B^-}(Y))$. In particular, the coefficient $P_{w,y,k}^{J,-1}$ of the parabolic Kazhdan-Lusztig polynomial $P_{w,y}^{J,-1}$ is non-negative and equal to the multiplicity of the irreducible Hodge module $\pi\mathbb{Q}_{Y^y}^H[-\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{Y^w}^H[-\ell(w)]$.

5 Finite-dimensional Schubert varieties

Set

$$(5.1) \quad X_w = Bw1_X = BwB/B \quad \text{for } w \in W.$$

Then we have the following result.

Proposition 5.1 (Kashiwara-Tanisaki [5]). *Set $X' = \bigcup_{w \in W} X_w$. Then X' is the flag manifold considered by Kac-Peterson [2], Tits [10], et al. In particular, we have the following.*

- (i) We have $X' = \bigsqcup_{w \in W} X_w$.
- (ii) For $w \in W$ X_w is a locally closed subscheme of X isomorphic to $\mathbb{A}^{\ell(w)}$.
- (iii) For $w \in W$ we have $\overline{X}_w = \bigsqcup_{y \in W, y \leq w} X_y$.

We call X_w for $w \in W$ a finite-dimensional Schubert cell and \overline{X}_w a finite-dimensional Schubert variety. Note that X' is not a scheme but an inductive limit of finite-dimensional projective schemes (an ind-scheme).

For $w \in W^J$, we set $Y_w = Bw1_Y = \pi(X_w)$. Similarly to Proposition 5.1 we have the following.

Proposition 5.2. *Set $Y' = \bigcup_{w \in W^J} Y_w$. Then we have the following.*

- (i) We have $Y' = \bigsqcup_{w \in W^J} Y_w$.
- (ii) For $w \in W^J$, Y_w is a locally closed subscheme of Y isomorphic to $\mathbb{A}^{\ell(w)}$.
- (iii) For $w \in W^J$, we have $\overline{Y}_w = \bigsqcup_{y \in W^J, y \leq w} Y_y$.
- (iv) For $w \in W^J$, we have $\pi^{-1}(Y_w) = \bigsqcup_{x \in W_J} X_{wx}$.

For a finite admissible subset Ω of W^J we set $Y'_\Omega = \bigcup_{w \in \Omega} Y'_w$. It is a finite dimensional projective scheme.

Let $\mathrm{HM}_d^B(Y'_\Omega)$ be the category of B -equivariant Hodge modules on Y'_Ω . For $w \in W^J$ the Hodge modules $\mathbb{Q}_{Y_w}^H[\ell(w)]$ and ${}^\pi\mathbb{Q}_{Y_w}^H[\ell(w)]$ are objects of $\mathrm{HM}_d^B(Y'_\Omega)$. Note that $\mathbb{Q}_{Y_w}[\ell(w)]$ is a perverse sheaf because Y_w is affine. Set

$$(5.2) \quad \mathrm{HM}_d^B(Y') = \varinjlim_{\Omega} \mathrm{HM}_d^B(Y'_\Omega), \quad K(\mathrm{HM}_d^B(Y')) = \varinjlim_{\Omega} K(\mathrm{HM}_d^B(Y'_\Omega)),$$

where Ω runs through finite admissible subsets of W^J . By the tensor product $K(\mathrm{HM}_d^B(Y'))$ is endowed with a structure of an R -module. Then any element of $K(\mathrm{HM}_d^B(Y'))$ is uniquely written as a finite sum in two ways

$$\sum_{w \in W^J} r_w [\mathbb{Q}_{Y_w}^H[\ell(w)]] \quad \text{and} \quad \sum_{w \in W^J} r'_w [{}^\pi\mathbb{Q}_{Y_w}^H[\ell(w)]] \quad \text{with } r_w, r'_w \in R.$$

Denote by $K(\mathrm{HM}_d^B(Y')) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_d^B(Y'))$ the involution of an abelian group induced by the duality functor \mathbb{D}_d . Then we have $\overline{r\overline{m}} = \overline{r} \overline{\overline{m}}$ for any $r \in R$ and $m \in K(\mathrm{HM}_d^B(Y'))$.

We can similarly define $\mathrm{HM}_d^B(X')$, $\mathbb{Q}_{X_w}^H[\ell(w)]$ and ${}^\pi\mathbb{Q}_{X_w}^H[\ell(w)]$ for $w \in W$, $K(\mathrm{HM}_d^B(X'))$, and $K(\mathrm{HM}_d^B(X')) \ni m \mapsto \overline{m} \in K(\mathrm{HM}_d^B(X'))$ (for $J = \emptyset$).

For $w \in W$ (resp. $w \in W^J$) we denote by $i_{X',w} : \text{pt} \rightarrow X'$ (resp. $i_{Y',w} : \text{pt} \rightarrow Y'$) denote the morphism with image $\{w1_X\}$ (resp. $\{w1_Y\}$). We define homomorphisms

$$(5.3) \quad \Psi : K(\text{HM}_d^B(X')) \rightarrow H, \quad \Psi^J : K(\text{HM}_d^B(Y')) \rightarrow H^{J,q}$$

of R -modules by

$$(5.4) \quad \Psi([M]) = \sum_{w \in W} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{X',w}^*(M)] \right) T_w,$$

$$(5.5) \quad \Psi^J([M]) = \sum_{w \in W^J} \left(\sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y',w}^*(M)] \right) T_w^{J,q}.$$

By the definition we have

$$(5.6) \quad \Psi([\mathbb{Q}_{X_w}^H[\ell(w)]]) = (-1)^{\ell(w)} T_w \quad \text{for } w \in W,$$

$$(5.7) \quad \Psi^J([\mathbb{Q}_{Y_w}^H[\ell(w)]]) = (-1)^{\ell(w)} T_w^{J,q} \quad \text{for } w \in W^J,$$

and hence Ψ and Ψ^J are isomorphisms.

Let $\pi' : X' \rightarrow Y'$ denote the projection. Let Ω be a finite admissible subset of W and set $\Omega' = \{w \in W^J; wW_J \cap \Omega \neq \emptyset\}$. Then Ω' is a finite admissible subset of W^J and π' induces a surjective projective morphism $X'_\Omega \rightarrow Y'_{\Omega'}$. Hence we can define a homomorphism $\pi'_! : K(HM^B(X')) \rightarrow K(HM^B(Y'))$ of R -modules by

$$(5.8) \quad \pi'_!([M]) = \sum_{k \in \mathbb{Z}} (-1)^k [H^k \pi'_!(M)].$$

Lemma 5.3. (i) *The following diagram is commutative.*

$$\begin{array}{ccc} K(\text{HM}_d^B(X')) & \xrightarrow{\Psi} & H \\ \pi'_! \downarrow & & \downarrow \varphi^{J,q} \\ K(\text{HM}_d^B(Y')) & \xrightarrow[\Psi^J]{} & H^{J,q} \end{array}$$

$$(ii) \quad \overline{\pi'_!(m)} = \pi'_!(\overline{m}) \text{ for any } m \in K(\text{HM}_d^B(X')).$$

$$(iii) \quad \overline{\Psi(m)} = \Psi(\overline{m}) \text{ for any } m \in K(\text{HM}_d^B(X')).$$

$$(iv) \quad \overline{\Psi^J(m)} = \Psi^J(\overline{m}) \text{ for any } m \in K(\text{HM}_d^B(Y')).$$

Proof. Let $w \in W^J$ and $x \in W_J$. Since $X_{wx} \rightarrow Y_w$ is an $\mathbb{A}^{\ell(x)}$ -bundle, we have $\pi'_!(\mathbb{Q}_{X_{wx}}^H) = \mathbb{Q}_{Y_w}^H[-2\ell(x)](-\ell(x))$, and hence

$$\pi'_!([\mathbb{Q}_{X_{wx}}^H[\ell(wx)]]) = (-q)^{\ell(x)}[\mathbb{Q}_{Y_w}^H[\ell(w)]].$$

Thus (i) follows from (5.6), (5.7) and (2.28).

The statement (ii) follows from the fact that π' is an inductive limit of projective morphisms and hence $\pi'_!$ commutes with the duality functor \mathbb{D}_d .

The statement (iii) is proved similarly to Kashiwara-Tanisaki [4], and we omit the details (see also Kazhdan-Lusztig [7]). Then the statement (iv) follows from (i), (ii), (iii), (2.24) and surjectivity of $\varphi^{J,q}$. \square

Theorem 5.4. *Let $w, y \in W^J$ such that $w \geq y$. Then we have*

$$H^{2k+1}i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H) = 0, \quad H^{2k}i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H) = \mathbb{Q}^H(-k)^{\oplus P_{y,w,k}^{J,q}}$$

for any $k \in \mathbb{Z}$. In particular, we have

$$\Psi^J([\pi\mathbb{Q}_{Y_w}^H[\ell(w)]]) = (-1)^{\ell(w)}C_w^{J,q}.$$

Proof. Let $w \in W^J$ and set

$$(-1)^{\ell(w)}\Psi^J([\pi\mathbb{Q}_{Y_w}^H[\ell(w)]]) = C = \sum_{y \in W^J, y \leq w} r_y T^{J,q}.$$

By the definition of $\pi\mathbb{Q}_{Y_w}^H[\ell(w)]$ we have $\mathbb{D}_d(\pi\mathbb{Q}_{Y_w}^H[\ell(w)]) = \pi\mathbb{Q}_{Y_w}^H\ell(w)$. Hence we obtain

$$(5.9) \quad \overline{C} = q^{-\ell(w)}C$$

by Lemma 5.3 (iv). By the definition of Ψ^J we have

$$(5.10) \quad r_y = \sum_{k \in \mathbb{Z}} (-1)^k [H^k i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H)],$$

and by the definition of $\pi\mathbb{Q}_{Y_w}^H[\ell(w)]$ we have

$$(5.11) \quad r_w = 1,$$

$$(5.12) \quad \text{for } y < w \text{ we have } H^k i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H) = 0 \text{ unless} \\ 0 \leq k \leq (\ell(w) - \ell(y) - 1).$$

Moreover, by the argument similar to Kazhdan-Lusztig [7] and Kashiwara-Tanisaki [4] we have

$$(5.13) \quad [H^k i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H)] \in R_k.$$

In particular, we have

$$(5.14) \quad \text{for } y < w \text{ we have } r_y \in \bigoplus_{k=0}^{\ell(w)-\ell(y)-1} R_k.$$

Thus we obtain $C = C_w^{J,q}$ by (5.9), (5.11), (5.14) and Proposition 2.3. Hence $r_y = P_{y,w}^{J,q}$. By (5.10) and (5.13) we have $[H^{2k+1}i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H)] = 0$ and $[H^{2k}i_{Y',y}^*(\pi\mathbb{Q}_{Y_w}^H)] = q^k P_{y,w,k}$ for any $k \in \mathbb{Z}$. The proof is complete. \square

We note that a result closely related to Theorem 5.4 above is already given in Deodhar [1].

By (2.40) and Theorem 5.4 we obtain the following.

Corollary 5.5. *We have*

$$[\mathbb{Q}_{Y'_w}^H[\ell(w)]] = \sum_{y \leq w} Q_{y,w}^{J,q} [\pi\mathbb{Q}_{Y'_y}^H[\ell(y)]]$$

in $K(\mathrm{HM}_d^B(Y'))$. In particular, the coefficient $Q_{y,w,k}^{J,q}$ of the inverse parabolic Kazhdan-Lusztig polynomial $Q_{y,w}^{J,q}$ is non-negative and equal to the multiplicity of the irreducible Hodge module ${}^\pi\mathbb{Q}_{Y'_y}^H[\ell(y)](-k)$ in the Jordan Hölder series of the Hodge module $\mathbb{Q}_{Y'_w}^H[\ell(w)]$.

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